ON CONTINUITY OF SOLUTION OF THE PROBLEM OF A REGULATOR ANALYTIC CONSTRUCTION IN SINGULAR PERTURBATIONS

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A linear problem of optimal control with quadratic quality criterion the differential constraints of which contain a small parameter accompanying some of the derivatives, is studied. It is shown that the solution of this problem with the small parameter tending to zero does itself tend to a solution of some problem of optimal control of the same class. The integrated term in the functional of this problem is constructed in a special manner and is not connected with the substitution of the root of the "degenerate" system into the functional. When the controlled objects are described in an exact manner, mathematical models of high order often ensue. Discarding certain small constants (time and mass constants, moments of inertia, etc.) yields equations of models of reduced order. A question arises of the correctness of such procedure in the sense of the nearness of the solutions of perturbed and unperturbed problems.

Let us consider the following problem of optimal control: to find a continuous r-dimensional vector function $u(t)$ which provides the functional

$$I(u) = \frac{1}{2}\left\|\begin{matrix} x(T) \\ z(T)\end{matrix}\right\|' \left\|\begin{matrix} F_1 & F_2 \\ F_2' & F_3\end{matrix}\right\| \left\|\begin{matrix} x(T) \\ z(T)\end{matrix}\right\| + \frac{1}{2}\int_0^T (x'D(t)x + u'R(t)u)\,dt \tag{1}$$

a minimum on the trajectories of the system

$$\begin{aligned} x^{\cdot} &= A_1(t)x + A_2(t)z + B_1(t)u, \quad x(0) = x_0, \quad x \in E^n \\ \lambda z^{\cdot} &= A_3(t)x + A_4(t)z + B_2(t)u, \quad z(0) = z_0, \quad z \in E^m \end{aligned} \tag{2}$$

Here $\lambda > 0$ is a small parameter, $T > 0$ is a fixed number, the prime denotes a transpose, all matrices are twice continuously differentiable on $[0, T]$, $R(t)$ is a positive-definite matrix and $D(t)$ is a positive-semi-definite matrix on $[0, T]$, and F is a constant, positive-definite matrix.

The system in (2) is singularly perturbed [1]. Voluminous literature exists dealing with singular perturbations in the problems of optimal control, and we mention [2, 6] which are allied to the subject of the present paper. In the majority of papers studying the singular perturbations in the problem of optimal control, the limiting problem is obtained from the initial problem when the perturbing parameter is equal to zero. Below we establish that the limiting problem for (1), (2) cannot be obtained at $\lambda = 0$, but must be constructed in a special manner.

Let us denote by $x(t, \lambda), z(t, \lambda)$ the optimal trajectories, $u(t, \lambda)$ the optimal control and by I_λ the minimal value of the functional in the problem (1),(2).

Theorem. Let

1) $\operatorname{rank}[B_2(T), A_4(T)B_2(T), \ldots, A_4^{m-1}(T)B_2(T)] = m$;

2) The real parts of the eigenvalues of the matrix $A_4(t)$ are negative for $t \in [0, T]$. Then for $\lambda \to 0$ we have

$x(t, \lambda) \to \bar{x}(t)$ uniformly in $t \in [0, T_1] \subset [0, T]$;
$z(t, \lambda) \to \bar{z}(t)$ uniformly in $t \in [T_0\ T_1] \subset [0, T]$;
$u(t, \lambda) \to \bar{u}(t)$ uniformly in $t \in [0, T_1] \subset [0, T]$; $I_\lambda \to I$

Here $\bar{x}(t)$ and $\bar{u}(t)$ denote the optimal trajectory and control, while I is the minimum value of the functional in the following problem of optimal control:

$$\bar{x}^{\cdot} = (A_1(t) - A_2(t)A_4^{-1}(t)A_3(t))\bar{x} + (B_1(t) - A_2(t)A_4^{-1}(t)B_2(t))\bar{u}, \quad (3)$$
$$\bar{x}(0) = x_0$$
$$I(\bar{u}) = \frac{1}{2}\bar{x}'(T)(F_1 - F_2F_3^{-1}F_2')\bar{x}(T) + \frac{1}{2}\int_0^T (\bar{x}'D(t)\bar{x} + \bar{u}'R(t)\bar{u})\,dt \quad (4)$$
$$\bar{z}(t) = -A_4^{-1}(t)[A_3(t)\bar{x}(t) + B_2(t)\bar{u}(t)]$$

Proof. We know [7] that the optimal control in the problem (1), (2) has the form

$$u(t, \lambda) = -R^{-1}\begin{Vmatrix} B_1 \\ \frac{1}{\lambda}B_2 \end{Vmatrix}' \begin{Vmatrix} K_1(t, \lambda) & \lambda K_2(t, \lambda) \\ \lambda K_2'(t, \lambda) & \lambda K_3(t, \lambda) \end{Vmatrix} \begin{Vmatrix} x(t, \lambda) \\ z(t, \lambda) \end{Vmatrix}$$

where the blocks $K_i(t, \lambda)$, $i = 1, 2, 3$ represent the solution of the following Cauchy problem (see [2, 3, 5, 6]):

$$dK_1/dt = -K_1A_1 - A_1'K_1 - K_2A_3 - A_3'K_2' + K_1S_1K_1 + K_1S_2K_2' + K_2S_2'K_1 + K_2S_3K_2' - D, \quad K_1(T) = F_1$$
$$\lambda\frac{dK_2}{dt} = -K_1A_2 - K_2A_4 - \lambda A_1'K_2 - A_3'K_3 + \lambda K_1S_1K_2 + K_1S_2K_3 + \lambda K_2S_2'K_2 + K_2S_3K_3, \quad K_2(T) = \frac{1}{\lambda}F_2$$
$$\lambda\frac{dK_3}{dt} = -\lambda K_2'A_2 - \lambda A_2'K_2 - K_3A_4 - A_4'K_3 + \lambda^2K_2'S_1K_2 + \lambda K_2'S_2K_3 + \lambda K_3S_2'K_2 + K_3S_3K_3, \quad K_3(T) = \frac{1}{\lambda}F_3$$
$$(S_1 = B_1R^{-1}B_1', \quad S_2 = B_1R^{-1}B_2', \quad S_3 = B_2R^{-1}B_2')$$

Similarly, for the problem (3), (4) we have

$$\bar{u}(t) = -R^{-1}\bar{B}'\bar{K}(t)\bar{x}(t)$$
$$\frac{d\bar{K}}{dt} = -\bar{K}\bar{A} - \bar{A}'\bar{K} + \bar{K}\bar{S}\bar{K} - D, \quad \bar{K}(T) = F_1 - F_2F_3^{-1}F_2'$$
$$(\bar{B} = B_1 - A_2A_4^{-1}B_2, \quad \bar{A} = A_1 - A_2A_4^{-1}A_3, \quad \bar{S} = \bar{B}R^{-1}\bar{B}')$$

Following the arguments in [6] and taking Condition 2 of the theorem into account, we can show that the following limiting passages take place uniformly in $t \in [0, T_1] \subset [0, T]$ as $\lambda \to 0$

$$
\begin{aligned}
&K_1(t,\lambda) \to \overline{K}_1(t) \equiv \overline{K}(t)\\
&K_2(t,\lambda) \to \overline{K}_2(t) \equiv -\overline{K}(t)A_2(t)A_4^{-1}(t)\\
&K_3(t,\lambda) \to \overline{K}_3(t) \equiv 0
\end{aligned}
\tag{5}
$$

To show this, it is sufficient to establish that

$$
\det\left\{\int_{-\infty}^{0} \exp(-A_4(T)s)\,S_3(T)\exp(-A_4'(T)s)\,ds\right\} \neq 0 \tag{6}
$$

The inequality (6) follows from Condition 1 of the theorem. Indeed, Condition 1 is equivalent [8] to the condition

$$
\operatorname{rank}\,[B_2(T),\ -A_4(T)B_2(T), \ldots, (-A_4(T))^{m-1}B_2(T)] = m
$$

and the last condition represents the criterion of complete controllability [8] of the autonomous system

$$
\frac{dy}{dt} = -A_4(T)\,y + B_2(T)\,v \tag{7}
$$

The second criterion of complete controllability of the system [7] is given by the condition

$$
\det\left\{\int_0^{\alpha} \exp(A_4(T)s)\,B_2(T)\,B_2'(T)\exp(A_4'(T)s)\,ds\right\} \neq 0 \tag{8}
$$

where α is a positive number. By virtue of the positive definiteness of $R(T)$ it follows from (8) that

$$
\det\left\{\int_0^{\alpha} \exp(A_4(T)s)\,S_3(T)\exp(A_4'(T)s)\,ds\right\} \neq 0
$$

Further, we have

$$
\begin{aligned}
&\int_{-\infty}^{0} \exp(-A_4(T)s)\,S_3(T)\exp(-A_4'(T)s)\,ds =\\
&\int_0^{\alpha} \exp(A_4(T)s)\,S_3(T)\exp(A_4'(T)s)\,ds +\\
&\int_{\alpha}^{\infty} \exp(A_4(T)s)\,S_3(T)\exp(A_4'(T)s)\,ds
\end{aligned}
$$

Now the Minkowski inequality implies the validity of the inequality (6) for the determinant of the sum of the positive-definite matrices [9].

The optimal trajectories $x(t,\lambda)$, $z(t,\lambda)$ are solutions of the problem

$$
\begin{aligned}
x^{\cdot} &= (A_1 - S_1K_1 - S_2K_2')x + (A_2 - \lambda S_1K_2 - S_2K_3)z, \quad x(0) = x_0\\
\lambda z^{\cdot} &= (A_3 - S_2'K_1 - S_3K_2')x + (A_4 - \lambda S_2'K_2 - S_3K_3)z, \quad z(0) = z_0
\end{aligned}
$$

Using the Tikhonov theorem [1] and taking into account (5) we find that when $\lambda \to 0$ $x(t, \lambda) \to \bar{x}_0(t)$ uniformly in $t \in [0, T_1] \subset [0, T]$, and $z(t, \lambda) \to \bar{z}_0(t)$ uniformly in $t \in [T_0, T_1] \subset [0, T]$ where

$$\bar{x}_0^{\cdot} = (A_1 - S_1\bar{K}_1 - S_2\bar{K}_2')\bar{x}_0 + A_2\bar{z}_0, \quad \bar{x}_0(0) = x_0$$
$$0 = (A_3 - S_2'\bar{K}_1 - S_3\bar{K}_2')\bar{x}_0 + A_4\bar{z}_0$$

Eliminating $\bar{z}_0$ from the above system and taking into account the expressions for $\bar{K}_1$ and $\bar{K}_2$ we find, that

$$\bar{x}_0^{\cdot} = (\bar{A} - \bar{S}\bar{K})\bar{x}_0, \quad \bar{x}_0(0) = x_0$$

i.e. $\bar{x}_0(t) \equiv \bar{x}(t)$. This completes the proof of the first assertion of the theorem.

We have

$$\bar{z}_0 = -A_4^{-1}(A_3 - S_2'\bar{K} + S_3(A_4')^{-1}A_2'\bar{K})\bar{x} = -A_4^{-1}[A_3\bar{x} - B_2R^{-1}(B_1' - B_2'(A_4')^{-1}A_2')\bar{K}\bar{x}] = -A_4^{-1}(A_3\bar{x} + B_2\bar{u}) = \bar{z}$$

and this proves the second assertion of the theorem.

The third assertion follows directly from the previous two, and from the form of the optimal controls $u(t, \lambda)$, $\bar{u}(t)$. We know that [8]

$$I_\lambda = \frac{1}{2}\begin{Vmatrix} x_0 \\ z_0 \end{Vmatrix}' \begin{Vmatrix} K_1(0,\lambda) & \lambda K_2(0,\lambda) \\ \lambda K_2'(0,\lambda) & \lambda K_3(0,\lambda) \end{Vmatrix} \begin{Vmatrix} x_0 \\ z_0 \end{Vmatrix}, \quad \bar{I} = \frac{1}{2} x_0'\bar{K}(0)x_0$$

and this implies the validity of the last assertion of the theorem.

Notes. 1°. Similar results can be obtained for a more general quadratic functional (1) using the arguments of [2-4, 6] and changing the corresponding conditions of the theorem.

2°. In the case $F_2 = 0, F_3 \neq 0$ the limiting problem can be obtained by substituting the root $\bar{z}$ into the system and $\bar{z}(T) = 0$ into the functional. If on the other hand F_2 and F_3 are zero matrices, the limiting problem will be obtained at $\lambda = 0$ from the initial problem as in [2,4].

REFERENCES

1. Vasil'eva, A.B. and Butuzov, V.F., Asymptotic Expansions of Solutions of Singularly Perturbed Equations. Moscow, "Nauka", 1973.

2. Kokotovic, P.V. and Yackel, R.A., Singular perturbations of linear regulators: Basic theorems. IEEE Trans. Automat. Contr., Vol. 17, № 1, 1972.

3. Yackel, R.A. and Kokotovic, P.V., A boundary layer method for the matrix Riccati equation. IEEE Trans. Automat. Contr., Vol. 18, № 1, 1973.

4. O'Malley, R.E. and Kung, C.F., The singularly perturbed linear state regulator problem. II. SIAM J. Control, Vol. 13, № 2, 1975.

5. Glizer, V.Ia. and Dmitriev, M.G., Singular perturbations in the linear problem of control with a quadratic functional. Differentsial'nye uravneniia, Vol. 11, № 11, 1975.

6. Glizer, V. Ia. and Dmitriev, M. G., Singular perturbations in the linear problem of optimal control with a quadratic functional. Dokl. Akad. Nauk SSSR, Vol. 225, № 5, 1975.
7. Kalman, R. E., Contributions to the theory of optimal control. Bol. Soc. mat. mexicana, Vol. 5, № 1, 1960.
8. Li, E. B. and Markus, L., Fundamentals of the Theory of Optimal Control. Moscow, "Nauka", 1969.
9. Bellman, R. E., Introduction to Matrix Analysis. N. Y., McGraw-Hill Book Co., 1960.

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